ON SPECIAL SUBVARIETIES OF THE TORELLI LOCUS

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In this talk, we will discuss the "Coleman-Oort" conjecture. Such conjecture dates back to the '80s. Initially formulated by Coleman, then it was endorsed by Oort in the '90s and it is still open, although several, partial results, also very recent, have been obtained in support of it. A very good historical account is given in [6].

We work over the complex numbers and we look at the Torelli locus $\mathcal{T}_g$ inside the moduli space $\mathcal{A}_g$ of g-dimensional principally polarized abelian varieties. The conjecture says that for genus g large enough there do not exist special (or Shimura) positive dimensional subvarieties $S \subset \mathcal{A}_g$ which are generically contained in the Torelli locus. Special subvarieties are totally geodesic with respect to the locally symmetric ambient geometry and contain CM points ([7, 5]).

It is known that for $g \leq 9$ the conjecture does not hold. Indeed, there are examples of special subvarieties of $\mathcal{T}_g$ (see [1, 6, 2, 4, 3]). They were obtained as families of Jacobians of Galois coverings $C \to C' := C/G$ with fixed monodromy, genera, number of branch points under a certain (sufficient) numerical condition, which we denote by $(*)$. These examples are 30+6, having respectively $g' := g(C') = 0, 1$, as presented in [2], respectively in [4].

First, we will describe how strong is the condition $(*)$. Indeed we will show that it bounds g' and, using this bound, we will prove that the 6 examples of [4] are the unique families of Galois coverings of curves of positive genus satisfying $(*)$. Then we will see that these families admit two fibrations in totally geodesic subvarieties, countably many of which are special (see [3]). This yields infinitely many new examples of positive dimensional special subvarieties of $\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$.

Finally, we move to a recent result shown in [9]. We will prove that the families of dihedral covers described in [8] determine two special subvarieties of $\mathcal{T}_2, \mathcal{T}_3$. In this way, we will trace out the first two examples of special subvarieties generically contained in the Torelli locus without using the condition $(*)$.

This is based on joint work with A. Ghigi and P. Frediani [3] and on [9].

References

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