DEEP LEARNING APPROXIMATION OF DIFFEOMORPHISMS VIA LINEAR-CONTROL SYSTEMS

ALESSANDRO SCAGLIOTTI

In this talk we will explore the interplay between Deep Learning and Control Theory. The starting point are the papers [2, 3], where it was independently observed that Residual Neural Networks (ResNets) can be naturally interpreted as discretization of continuous-time control systems. In this framework, it is possible to show that the expressivity of a ResNet is strictly related to controllability properties of the underlying control system. In this regards, in [1] the authors considered a linear-control system in $\mathbb{R}^n$ of the form

$$\dot{x}(t) = \sum_{i=1}^{k} F_i(x(t))u_i(t), \qquad t \in [0,1], \tag{1}$$

where $F_1, \ldots, F_k : \mathbb{R}^n \to \mathbb{R}^n$ are regular controlled vector fields, and $u_1, \ldots, u_k \in L^2([0,1], \mathbb{R})$ are the admissible controls. For every $u = (u_1, \ldots, u_k) \in L^2([0,1], \mathbb{R}^k)$ we can consider the diffeomorphism $\Phi_u : \mathbb{R}^n \to \mathbb{R}^n$ defined as the evaluation at the final instant of the flow induced by (1), i.e.,

$$\Phi_u(x_0) := x^u_{x_0}(1)$$

for every $x_0 \in \mathbb{R}^n$, where $x^u_{x_0} : [0,1] \to \mathbb{R}^n$ is the solution of (1) corresponding to the control $u = (u_1, \ldots, u_k)$ and to the initial datum $x^u_{x_0}(0) = x_0$. In [1], under mild hypotheses on the controlled vector fields $F_1, \ldots, F_k$, it was proved that, given a diffeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ isotopic to the identity and a compact set $K \subset \mathbb{R}^n$, for every $\varepsilon > 0$ there exists a control $u \in L^2([0,1], \mathbb{R}^k)$ such that

$$||\Psi - \Phi_u||_{C^0,K} \leq \varepsilon.$$

Starting from this theoretical result, in [4] we introduced an optimal control problem to model this approximation task. Indeed, following a data-driven approach, we imagined to observe the action of the diffeomorphism Ψ on an ensemble of training points $\{x_1, \ldots, x_N\}$ with $N \geq 1$. Therefore, we considered the following minimization problem on the space of admissible controls:

$$\frac{1}{N} \sum_{j=1}^{N} |\Psi(x_j) - \Phi_u(x_j)|_2^2 + \frac{\beta}{2} ||u||^2_{L^2} \to \min, \tag{2}$$

where $\beta > 0$ is a hyper-parameter that tunes the L^2-norm regularization. The discretization of (1) leads to a ResNet, and the numerical resolution of (2) can be seen as the training process for the ResNet just obtained. We explore two possible training strategies. The first one consists in projecting onto a finite-dimensional subspace of $L^2([0,1], \mathbb{R}^k)$ the gradient flow induced by the functional (2). The second one relies on an iterative algorithm based on the Pontryagin Maximum Principle. Finally, we provide an estimate for the generalization error by means of a Γ-convergence result.

References

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SISSA-Trieste

Email address: ascaglio@sissa.it