GEOMETRY OF 1-CODIMENSIONAL MEASURES IN THE HEISENBERG GROUPS

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In the Euclidean spaces the notion of rectifiability of a measure is linked to the metric by the celebrated:

Theorem 1 (Preiss, [1]). *Suppose $0 \leq m \leq n$ are integers, ϕ is a Radon measure on $\mathbb{R}^n$ and:*

$$0 < \Theta^m(\phi, x) := \lim_{r\to 0} \frac{\phi(U_r(x))}{r^m} < \infty \qquad \text{at } \phi\text{-almost every } x, \tag{1}$$

where $U_r(x)$ is the Euclidean ball of centre x and radious r. Then ϕ is m-rectifiable, i.e., ϕ-almost all of $\mathbb{R}^n$ can be covered by countably many m-dimensional Lipschtitz submanifolds of $\mathbb{R}^n$.

The most difficult part of the proof of Theorem 1 is to show that the existence of the density, namely that (1) holds, implies that the measure ϕ has flat tangents, i.e.:

$$\mathrm{Tan}(\phi, x) \subseteq \Theta^m(\phi, x)\{\mathcal{H}^m \llcorner V : V \text{ is an } m\text{-plane}\} \qquad \text{at } \phi\text{-almost every point.} \tag{2}$$

The fact that the inclusion (2) implies Theorem 1 is a consequence of the Marstrand-Mattila rectifiability criterion, see for instance [1, Corollary 5.4]. The proof of such inclusion depends on the structure of the Euclidean ball and it is not known whether it is possible to extend it to a general finite dimensional Banach space. The only progress in this direction, to our knowledge, was achieved by A. Lorent, who proved that 2-locally uniform measures in ℓ^3_∞ are rectifiable, see Theorem 5 in [4].

In this talk, I will give present the first extension of Theorem 1 outside Euclidean spaces:

Theorem 2 ([5, 6]). *Suppose ϕ is a Radon measure in $\mathbb{H}^n$ such that:*

$$0 < \Theta^{2n+1}(\phi, x) := \lim_{r\to 0} \frac{\phi(B_r(x))}{r^{2n+1}} < \infty \qquad \text{for } \phi\text{-a.e. } x, \tag{3}$$

where $B_r(x)$ is the Koranyi ball. Then $\mathbb{H}^n$ can be covered ϕ-almost all by countably many $C^1_{\mathbb{H}^n}$-regular surfaces, which are smooth surfaces in a very intrinsic sense and were introduced in [3] *and are fractals from the Euclidean point of view* [2].

References

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