IRREDUCIBLE GENERALIZED NUMERICAL SEMIGROUPS AND A GENERALIZATION OF WILF'S CONJECTURE

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Let $\mathbb{N}$ be the set of non negative integers and d be a positive integer. A *generalized numerical semigroup*(GNS) is a submonoid S of $\mathbb{N}^d$ such that the set $\mathrm{H}(S) = \mathbb{N}^d \setminus S$ is finite. The set $\mathrm{H}(S)$ is called the set of *gaps* (or *holes*) of S, its cardinality $\mathrm{g}(S) = |\,\mathrm{H}(S)|$ is called the *genus* of S. Generalized numerical semigroups have been introduced in [5] as a straightforward generalization of the well known concept of numerical semigroup, that is a submonoid of $\mathbb{N}$ having finite complement in it. Some notions and results concerning numerical semigroups have been generalized to the context of GNSs in [3] and in other recent papers. One of the most important of these generalizations concerns with the concept of irreducibility. A GNS is said to be *irreducible* if it cannot be expressed as the intersection of two generalized numerical semigroups properly containing it. We prove that such GNSs are characterized by the set $\mathrm{SG}(S) = \{\mathbf{x} \in \mathrm{H}(S) \mid 2\mathbf{x} \in S, \mathbf{x} + \mathbf{s} \in S \text{ for all } \mathbf{s} \in S \setminus \{\mathbf{0}\}\}$, called the set of *special gaps* of S. We show in particular that S is irreducible if and only if $|\,\mathrm{SG}(S)| = 1$. This result is the starting point to obtain other properties and characterizations. For instance, if $S \subseteq \mathbb{N}^d$ is an irreducible GNS and $\mathrm{SG}(S) = \{\mathbf{f}\}$ then it is verified that $\mathbf{f}$ is the maximum in the set $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$. This fact allows to address a problem posed in [5], that is to define the analogue of the *Frobenius number* of a numerical semigroup for a GNS. Recalling that the Frobenius number of a numerical semigroup $S \subseteq \mathbb{N}$ is $\mathrm{F}(S) = \max(\mathrm{H}(S))$ (with the convention $\mathrm{F}(\mathbb{N}) = -1$), in [5] a *Frobenius element* of a GNS is defined up to a particular class of total orders in $\mathbb{N}^d$, called *relaxed monomial orders*. In particular, if $\prec$ is such an order on $\mathbb{N}^d$, the *Frobenius element* of a GNS $S \subseteq \mathbb{N}^d$ is defined as $\mathrm{F}_\prec(S) = \max_\prec(\mathrm{H}(S))$. Our results show that for irreducible generalized numerical semigroups it is verified that $\mathrm{F}_\prec(S) = \mathbf{f}$ for every relaxed monomial order $\prec$, where $\mathbf{f}$ is the unique element in $\mathrm{SG}(S)$. Another nice characterization for irreducible GNSs shows that a GNS $S \subseteq \mathbb{N}^d$ is irreducible if and only if there exists $\mathbf{f} = (f_1, \dots, f_d) \in \mathrm{H}(S)$ such that $\prod_{i=1}^d (f_i + 1) = 2\,\mathrm{g}(S)$ or $\prod_{i=1}^d (f_i + 1) = 2\,\mathrm{g}(S) - 1$. Such equalities inspired us the formulation of a generalization in the context of GNSs of a well known conjecture for numerical semigroups, called *Wilf's conjecture* (see [4] for a survey on the argument). Such a conjecture is still an open (and intriguing) problem for numerical semigroups, even if it is known that it is true for a large number of classes of numerical semigroups. After defining our generalized version of the conjecture, we prove that it is true for all irreducible GNSs and some other classes of GNSs.

References

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